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Geometric properties of Dirac fields in a Riemannian space–time: II

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Abstract. Our classification of Einstein–Dirac fields is extended to fields for which the principal spinors are independent of each other. It is shown that, when the Ricci scalar vanishes, the properties of the field are, to a large extent, determined by the twists of the two null congruences defined by the field.

1. Introduction

In the previous paper (Radford and Klotz 1979, hereafter referred to as I) we discussed the classification of Dirac–Einstein fields, that is solutions of Dirac equations in a Riemannian space in which Einstein's gravitational field equations are valid, and the structure of one class of such fields (type-II fields). We now propose to consider, in a similar way, Dirac fields of type I. The reader is referred to I for details of the classification itself, notational and mathematical preliminaries connected with the 2-spinor and Newman–Penrose (NP) formalism which we employ again here, and the full bibliography of the subject.

A type-I field occurs when the two Dirac spinors u_A and v_A are independent of each other (they are proportional for a type-II field). In this case, we can choose the spinor dyad o_A, l_A in such a way that

$$u_A = fo_A \qquad \text{and} \qquad v_A = -gl_A \tag{1}$$

for some complex functions f and g of the coordinates. Thus we have

$$u_0 = v_1 = 0 \qquad u_1 = f \qquad v_0 = g. \tag{2}$$

Since the directions l^α and n^α of the NP tetrad are now fixed, the remaining tetrad freedom is one of boosts in the l^α–n^α plane and of rotations in the m^α–$\bar{m}^\alpha$ plane. With A and ϕ as real functions, this freedom may be expressed as

$$\begin{aligned} l^\alpha &\to \hat{l}^\alpha = A^{-1} l^\alpha \\ m^\alpha &\to \hat{m}^\alpha = e^{i\phi} m^\alpha \\ \bar{m}^\alpha &\to \hat{\bar{m}}^\alpha = e^{-i\phi} \bar{m}^\alpha \\ n^\alpha &\to \hat{n}^\alpha = A n^\alpha. \end{aligned} \tag{3}$$

In terms of the spinor dyad we can also write

$$\begin{pmatrix} o^A \\ \iota^A \end{pmatrix} \to \pm \begin{pmatrix} A^{-1/2}\,e^{i\phi/2} & 0 \\ 0 & A^{1/2}\,e^{-i\phi/2} \end{pmatrix} \begin{pmatrix} o^A \\ \iota^A \end{pmatrix}. \tag{4}$$

We can now write down the Dirac equations and the conserved current, energy–momentum and the s^α vector (which is analogous to the 'spin density' vector of a classical fluid with 'spin').

2. Dirac equations, conserved entities and integrability conditions

With the choice (1) of the spinor directions, the Dirac equations, using the NP spin coefficients (see I for detailed definition), become

$$Df = (\rho - \epsilon)f + (i/\sqrt{2}L)\bar{g} \tag{5a}$$

$$\delta f = (\tau - \beta)f \tag{5b}$$

and

$$\Delta g = (\gamma - \mu)g - (i/\sqrt{2}L)\bar{f} \tag{5c}$$

$$\bar{\delta} g = (\alpha - \pi)g. \tag{5d}$$

Similarly, the conserved vectors are

$$j^\alpha = f\bar{f}l^\alpha + g\bar{g}n^\alpha \tag{6}$$

(that is, the necessarily real current density) and

$$s^\alpha = -f\bar{g}m^\alpha \tag{7}$$

(which is complex). From I, the energy–momentum tensor is given by

$$\begin{aligned} 4KT_{\alpha\beta} = {} & \phi_{00}n_\alpha n_\beta - 2\phi_{01}n_{(\alpha}\bar{m}_{\beta)} - 2\phi_{10}n_{(\alpha}m_{\beta)} + \phi_{02}\bar{m}_\alpha\bar{m}_\beta + \phi_{20}m_\alpha m_\beta + \phi(4l_{(\alpha}n_{\beta)} - g_{\alpha\beta}) \\ & - 2\phi_{12}l_{(\alpha}\bar{m}_{\beta)} - 2\phi_{21}l_{(\alpha}m_{\beta)} + \phi_{22}l_\alpha l_\beta + 12\Lambda l_{(\alpha}n_{\beta)} \end{aligned} \tag{8}$$

where

$$\begin{aligned} & \Lambda(=24R) = K(fg + \bar{f}\bar{g})/3\sqrt{2}L \\ & \phi_{00} = 2iK[\bar{g}Dg - gD\bar{g} + (\bar{\epsilon} - \epsilon)g\bar{g}] \\ & \phi_{01} \equiv \bar{\phi}_{10} = iK[\bar{g}\delta g + (2\bar{\pi} - \beta)g\bar{g} - \kappa f\bar{f}] \\ & \phi_{02} \equiv \bar{\phi}_{20} = 2iK(\bar{\lambda}g\bar{g} - \sigma f\bar{f}) \\ & \phi \equiv \phi_{11} - 3\Lambda = iK[(\bar{\rho} - \rho)f\bar{f} + (\bar{\mu} - \mu)g\bar{g}] \\ & \phi_{12} \equiv \bar{\phi}_{21} = iK[f\delta\bar{f} + (\bar{\alpha} - 2\tau)f\bar{f} + \bar{\nu}g\bar{g}] \\ & \phi_{22} = 2iK[f\Delta\bar{f} - \bar{f}\Delta f + (\bar{\gamma} - \gamma)f\bar{f}]. \end{aligned} \tag{9}$$

In obtaining these expressions we made use of the Dirac equations (5).

As in the case of type-II fields, we obtain integrability conditions of the Dirac system using the NP commutators. First, we apply the commutator

$$\delta D - D\delta = (\bar{\alpha} + \beta - \bar{\pi})D + \kappa\Delta - \sigma\bar{\delta} - (\bar{\rho} + \epsilon - \bar{\epsilon})\delta \tag{10}$$

to f. Using the Dirac equations (5) and the three NP relations (Flaherty 1976)

$$\begin{aligned} D\beta-\delta\epsilon &= (\alpha+\pi)\sigma+(\bar\rho-\bar\epsilon)\beta-(\mu+\gamma)\kappa-(\bar\alpha-\bar\pi)\epsilon+\psi_1 \\ D\tau-\Delta\kappa &= (\tau+\bar\pi)\rho+(\bar\tau+\rho)\sigma+(\epsilon-\bar\epsilon)\tau-(3\gamma+\bar\gamma)\kappa+\psi_1+\phi_{01} \\ \delta\rho-\bar\delta\sigma &= \rho(\bar\alpha+\beta)-\sigma(3\alpha-\bar\beta)+(\rho-\bar\rho)\tau+(\mu-\bar\mu)\kappa-\psi_1+\phi_{01} \end{aligned} \tag{11}$$

we readily obtain

$$\Delta(\kappa f)-\bar\delta(\sigma f)=-(\mathrm{i}/\sqrt{2}L)\tau\bar g+f[\kappa(2\gamma+\bar\gamma-\bar\mu)+\sigma(\bar\beta-\bar\tau-2\alpha)-\psi_1]. \tag{12}$$

Similarly, applying to g the commutator

$$\bar\delta\Delta-\Delta\bar\delta=-\nu D+(\bar\tau-\alpha-\bar\beta)\Delta+(\bar\mu-\gamma+\bar\gamma)\bar\delta+\lambda\delta, \tag{13}$$

and using (Flaherty 1976)

$$\begin{aligned} D\nu-\Delta\pi &= \mu(\pi+\bar\tau)+(\bar\pi+\tau)\lambda+(\gamma-\bar\gamma)\pi-(3\epsilon+\bar\epsilon)\nu+\psi_3+\phi_{21} \\ \delta\lambda-\bar\delta\mu &= (\rho-\bar\rho)\nu+(\mu-\bar\mu)\pi+(\alpha+\bar\beta)\mu+(\bar\alpha-3\beta)\lambda-\psi_3+\phi_{21} \\ \Delta\alpha-\bar\delta\gamma &= (\rho+\epsilon)\nu-(\tau+\beta)\lambda+(\bar\gamma-\bar\mu)\alpha+(\bar\beta-\bar\tau)\gamma-\psi_3, \end{aligned} \tag{14}$$

we find

$$\delta(\lambda g)-D(\nu g)=-(\mathrm{i}/\sqrt{2}L)\pi\bar f+g[\nu(2\epsilon+\bar\epsilon-\bar\rho)+\lambda(\bar\alpha-2\beta-\bar\pi)-\psi_3]. \tag{15}$$

Here ψ_1, ψ_3 are two of the five tetrad components of the Weyl tensor (Flaherty 1976). These conditions must be satisfied by any solution f, g of the Dirac field equations (5).

3. Vacuum space–time test solutions and subclassification of type-I fields

Let us consider first the null directions l^α and n^α, defined by a type-I field, to be geodesic and shear-free. These geometrical conditions are expressed by

$$\kappa=\sigma=\lambda=\nu=0, \tag{16}$$

and then

$$f\psi_1=-(\mathrm{i}/\sqrt{2}L)\tau\bar g \tag{17}$$

and

$$g\psi_3=-(\mathrm{i}\sqrt{2}L)\pi\bar f. \tag{18}$$

In a type-D vacuum space–time the above null directions coincide with the two geodesic and shear-free directions of the Weyl tensor. For test solutions on such space–time we now have the following theorem

Theorem 1. The only type-D vacuum space–times which admit test Dirac fields of type I, in which each of the null directions defined by the Dirac field is geodesic and shear-free, are NUT and related space–times.

We may note that this result is much more restrictive than the corresponding result for a field with zero rest mass.

Let us now subdivide type-I fields as follows (as we have subdivided type-II fields in I).

Definition. A Dirac field of type I will be said to be
(i) of type I_G if

$$\Lambda = fg + \bar{f}\bar{g} \neq 0$$

and
(ii) of type I_T if

$$\Lambda = 0.$$

Thus, for a I_G field the trace of the energy–momentum tensor is not allowed to vanish, while it vanishes for a I_T field. We have stated the Wainwright (1971) energy conditions for Dirac fields in I and we can now consider their geometrical implications for type-I fields. Actually, we shall only prove the results for I_T fields, although similar conditions can be derived for I_G fields also. It turns out that, in the latter case, the restrictions imposed on the field by the energy conditions are algebraically involved and lack any apparent geometric interpretation.

For a type-I_T field, the condition $\Lambda = 0$ implies that $fg = -\bar{f}\bar{g}$, so that

$$g = \mathrm{i}h\bar{f} \tag{19}$$

where h is a real, non-vanishing function. From the equations (5) we get

$$\delta h = (\bar{\alpha} + \beta - \tau - \bar{\pi})h \tag{20}$$

and

$$\begin{aligned}
\phi_{00} &= -4Kh^2 f\bar{f}\omega_\rho \\
\phi_{01} &= h^2\phi_{12} + \mathrm{i}K[h^2(\tau + \bar{\pi}) - h^4\bar{\nu} - \kappa]f\bar{f} \\
\phi_{02} &= 2\mathrm{i}K(\bar{\lambda}h^2 - \sigma)f\bar{f} \\
\phi &= 2K(\omega_\mu h^2 - \omega_\rho)f\bar{f} \\
\phi_{12} &= \mathrm{i}K[\delta\bar{f} + (\bar{\alpha} - 2\tau)\bar{f} + \nu h^2\bar{f}]f \\
\phi_{22} &= 4Kf\bar{f}\omega_\mu
\end{aligned} \tag{21}$$

where

$$\omega_\rho = \tfrac{1}{2}\mathrm{i}(\rho - \bar{\rho}) \tag{22}$$

is the twist of the l_α congruence, while

$$\omega_\mu = \tfrac{1}{2}\mathrm{i}(\bar{\mu} - \mu) \tag{23}$$

is the twist of the n_α congruence.

4. Type-I_T fields and the energy conditions

It is clear from equation (21) that conditions which can be imposed on the energy–momentum tensor (Wainwright 1971; see also I) can be interpreted in terms of the twist of null congruences. We have the following theorem.

Theorem 2. For a Dirac field of type $\mathrm{I_T}$ and energy class $\mathrm{E_1}$ at least one of the two null congruences defined by the field must have a non-zero twist; if both twists are non-zero, then

$$\omega_\mu \omega_\rho < 0,$$

so that they are of opposite sign.

Proof. Since the velocity of an arbitrary observer can be written as

$$u_\alpha = p l_\alpha + q n_\alpha + s m_\alpha + \bar{s} \bar{m}_\alpha \tag{24}$$

(where $pq - s\bar{s} = \frac{1}{2}$ so that $u_\alpha u^\alpha = 1$), we have (in general)

$$\begin{aligned} E(u) = \frac{1}{4Kp^2}\{&p^4 \phi_{00} + 2p^3(s\phi_{01} + \bar{s}\phi_{10}) \\ &+ p^2[(1+2s\bar{s})(\phi + 6\Lambda) + 2s\bar{s}\phi + s^2\phi_{02} + \bar{s}^2\phi_{20}] \\ &+ p(1+2s\bar{s})(s\phi_{12} + \bar{s}\phi_{21}) + \tfrac{1}{4}(1+2s\bar{s})^2\phi_{22}\}. \end{aligned} \tag{25}$$

For the field to be of class $\mathrm{E_1}$ we require $E(u) \neq 0$, i.e. $E(u) = 0$ should admit only complex or zero solutions for p (for all s); in the present case (type-$\mathrm{I_T}$ field) for an observer with $s = 0$,

$$E(u)|_{s=0} = -f\bar{f}[p^4 h^2 \omega_\rho + p^2 \tfrac{1}{2}(\omega_\rho - \omega_\mu h^2) - \tfrac{1}{4}\omega_\mu]/p^2. \tag{25'}$$

Clearly, at least one of ω_ρ or ω_μ must be non-zero (otherwise, $E(u) = 0$ for all observers with $s = 0$); if $\omega_\rho \neq 0 \neq \omega_\mu$, then the solution of $E(u)|_{s=0} = 0$ are

$$p^2 = \tfrac{1}{2}(\omega_\mu/\omega_\rho), \qquad -\tfrac{1}{2}(f\bar{f}/g\bar{g})$$

and give only complex solutions for p, provided $\omega_\mu \omega_\rho < 0$.

Theorem 3. If the twist of one of the congruences vanishes (say, $\omega_\rho = 0$, $\omega_\mu \neq 0$), then the field $\mathrm{I_T}$ is of class $\mathrm{E_1}$ if and only if there exists a null tetrad in which the energy–momentum tensor takes the form

$$\begin{aligned} T_{\alpha\beta} = f\bar{f}[&\omega_\mu l_\alpha l_\beta + \tfrac{1}{2}h^2\omega_\mu(4l_{(\alpha}n_{\beta)} - g_{\alpha\beta}) + \tfrac{1}{2}\mathrm{i}(\bar{\sigma} - \lambda h^2)m_\alpha m_\beta + \tfrac{1}{2}\mathrm{i}(\bar{\lambda}h^2 - \sigma)\bar{m}_\alpha \bar{m}_\beta \\ &- \tfrac{1}{2}\mathrm{i}(\bar{\tau} + \pi - \nu h^2 - \kappa/h^2)l_{(\alpha}m_{\beta)} + \tfrac{1}{2}\mathrm{i}(\tau + \bar{\pi} - \bar{\nu}h^2 - \bar{\kappa}/h^2)l_{(\alpha}\bar{m}_{\beta)}] \end{aligned} \tag{27}$$

with the twist restricted by

$$|\omega_\mu|^3 - \left(\frac{|a|^2 + 2|b|^2 h^2}{h^4 (f\bar{f})^2}\right)|\omega_\mu| + 2\epsilon |a|\,|b|^2 \cos(2\zeta - \psi) \geq 0 \tag{28}$$

where

$$b = \phi_{12}/4K = |b|\,\mathrm{e}^{\mathrm{i}\zeta}$$

$$a = \phi_{02}/4K = |a|\,\mathrm{e}^{\mathrm{i}\psi}$$

$$s = |s|\,\mathrm{e}^{\mathrm{i}\theta}$$

$$\omega_\mu = \epsilon|\omega_\mu| \qquad \epsilon = \pm 1.$$

Proof. When $\omega_\rho = \Lambda = 0$, we have (from equation (25))

$$4Kp^2E(u) = 2p^3(s\phi_{01} + \bar{s}\phi_{10}) + p^2[(1+4s\bar{s})\phi + s^2\phi_{02} + \bar{s}^2\phi_{20}]$$
$$+ p(1+2s\bar{s})(s\phi_{12} + \bar{s}\phi_{21}) + \tfrac{1}{4}(1+2s\bar{s})^2\phi_{22}.$$

Thus $E(u) = 0$ is a cubic which will have no real roots (coefficient of $p^3 = 0$) if and only if $\phi_{01} = 0$ and

$$(s\phi_{12} + \bar{s}\phi_{21})^2 < \phi_{22}[(1+4s\bar{s})\phi + s^2\phi_{02} + \bar{s}^2\phi_{20}]$$

for all s. The first condition, $\phi_{01} = 0$, gives the required form of the energy–momentum tensor (using equation (21)) and, as $|s|$ can be made arbitrarily large, the second condition becomes (upon making the appropriate substitutions)

$$2|b|^2\cos^2(\theta+\zeta) - (f\bar{f})^2h^2\omega_\mu^2 - f\bar{f}|a|\omega_\mu\cos(2\theta+\psi) \leqslant 0.$$

Choosing θ to maximise the left-hand side of this expression then gives (28), completing the proof.

Finally, we recall (see I) that to satisfy the strong energy condition the field must be of class E_2 and satisfy $E(u) > 0$. Using (25′) we immediately deduce the following theorem.

Theorem 4. For a Dirac field of type I_T satisfying the strong energy condition, the twist of l_α congruence (tangent to $u_A\bar{u}_A$) must be non-positive and that of n_α congruence (tangent to $v_A\bar{v}_A$) non-negative, with at least one of the twists non-zero.

Thus, one of the three cases

(i) $\omega_\rho = 0$, $\omega_\mu > 0$,
(ii) $\omega_\mu = 0$, $\omega_\rho < 0$ and
(iii) $\omega_\rho < 0$, $\omega_\mu > 0$

must hold.

5. Conclusions

The usefulness of the results obtained in this paper and in I depends, of course, on the existence of solutions of the Einstein–Dirac system of field equations. Although many test solutions are known, this is not always the case if we seek exact solutions of the coupled equations. For example, there does not appear to be a spherically symmetric, static solution (with static current vector, j^α). This will be shown elsewhere with some examples of cases when complete solutions do exist. We may observe that the theorems proved for a test Dirac field are more restrictive than the corresponding result for zero-rest-mass fields. Thus, with reference to theorem 1, all type-D vacuum space-times admit test solutions of the source-free Maxwell equations in which the two principal null directions of the field tensor are geodesic and shear-free. It is interesting to note also the important role played by the twist in the type-I_T case, the main case discussed in this article.

Theorem 4 itself is rather reminiscent of the conditions on the helicity for positive and negative energy solutions of the zero-rest-mass field equations in flat space–time (see Penrose 1975). For the other type-I fields (type I_G) the restrictions, imposed by the energy conditions, become algebraically very involved and seem to lack any immediate geometrical interpretation, and for this reason are not discussed in the present paper.

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